On the black-box complexity of Sperner's Lemma

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Abstract

We present several results on the complexity of various forms of Sperner's Lemma in the black-box model of computing. We give a deterministic algorithm for Sperner problems over pseudo-manifolds of arbitrary dimension. The query complexity of our algorithm is linear in the separation number of the skeleton graph of the manifold and the size of its boundary. As a corollary we get an $O(\sqrt{n})$ deterministic query algorithm for the black-box version of the problem **2D-SPERNER**, a well studied member of Papadimitriou's complexity class PPAD. This upper bound matches the $\Omega(\sqrt{n})$ deterministic lower bound of Crescenzi and Silvestri. The tightness of this bound was not known before. In another result we prove for the same problem an $\Omega(\sqrt[4]{n})$ lower bound for its probabilistic, and an $\Omega(\sqrt[8]{n})$ lower bound for its quantum query complexity, showing that all these measures are polynomially related.

Classification: computational and structural complexity, quantum computation and information.

1 Introduction

Papadimitriou defined in [17, 18] the complexity classes PPA, PPAD, and PSK in order to classify total search problems which have always a solution.The class PSK was renamed PPADS in [5]. These classes can be characterized by some underlying combinatorial principles. The class Polynomial Parity Argument (PPA) is the class of NP search problems, where the existence of the solution is guaranteed by the fact that in every finite graph the number of vertices with odd degree is even. The class PPAD is the directed version of PPA, and its basic search problem is the following: in a directed graph, where the in-degree and the out-degree of every vertex is at most one, given a source, find another source or a sink. In the class PPADS the basic search problem is more restricted than in PPAD: given a source, find a sink.

These classes are in fact subfamilies of TFNP, the family of all total NP-search problems, introduced by Megiddo and Papadimitriou [16]. Other important subclasses of TFNP are Polynomial Pigeonhole Principle (PPP) and Polynomial Local Search (PLS). The elements of PPP are problems which by their combinatorial nature obey the pigeonhole principle and therefore have a solution. In a PLS problem, one is looking for a local optimum for a particular objective function, in some neighborhood structure. All these classes are interesting because they contain search problems not known to be solvable in polynomial time, but which are also somewhat easy in the sense that they can not be NP-hard unless NP = co-NP.

Another point that makes the parity argument classes interesting is that there are several natural problems from different branches of mathematics that belong to them. For example, in a graph with odd degrees, when a Hamiltonian path is given, a theorem of Smith [25] ensures that there is another Hamiltonian path. It turns out that finding this second path belongs to the class PPA [18]. A search problem coming from a modulo 2 version of Chevalley's theorem [18] from number theory is also in PPA. Complete problems in PPAD are the search versions of Brouwer's fixed point theorem, Kakutani's fixed point theorem, Borsuk-Ulam theorem, and Nash equilibrium (see [18]).

The classical Sperner's Lemma [22] states that in a triangle with a regular triangulation whose vertices are labeled with three colors, there is always a trichromatic triangle. This lemma is of special interest since some customary proofs for the above topological fixed point theorems rely on its combinatorial content. However, it is unknown whether the corresponding search problem, that Papadimitriou [18] calls **2D-SPERNER**, is complete in PPAD. Variants of Sperner's Lemma also give rise to other problems in the parity argument classes. Papadimitriou [18] has proven that a 3-dimensional analogue of **2D-SPERNER** is in fact complete in PPAD.

The study of query complexities of the black-box versions of several problems in TFNP is an active field of research. Several recent results point into the direction that quantum algorithms can give only a limited speedup over deterministic ones in this framework. The collision lower bound of Aaronson [1] and Shi [20] about PPP, and the recent result of Santha and Szegedy [19] on PLS imply that the respective deterministic and quantum complexities are polynomially related. As a consequence, if an efficient quantum algorithm exists for a problem in these classes, it must exploit its specific structure. In a related issue, Buresh-Oppenheim and Morioka [8] have obtained relative separation results among PLS and the polynomial parity argument classes.

2   Results

A *black-box problem* is a relation $R \subseteq S \times T$ where T is a finite set and $S \subseteq \Sigma^n$ for some finite set Σ. The oracle input is a function $x \in S$, hidden by a black-box, such that x_i, for $i \in \{1, \ldots, n\}$ can be accessed via a query parameterized by i. The output of the problem is some $y \in T$ such that $(x, y) \in R$. A special case is the *functional oracle problem* when the relation is given by a function $A : S \to T$, the (unique) output is then $A(x)$. We say that A is *total* if $S = \Sigma^n$.

In the query model of computation each query adds one to the complexity of the algorithm, but all other computations are free. The state of the computation is represented by three registers, the query register $i \in \{1, \ldots, n\}$, the answer register $a \in \Sigma$, and the work register z. The computation takes place in the vector space spanned by all basis states $|i\rangle|a\rangle|z\rangle$. In the *quantum query model* introduced by Beals, Buhrman, Cleve, Mosca and de Wolf [4] the state of the computation is a complex combination of all basis states which has unit length in the norm l_2. In the randomized

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model it is a non-negative real combination of unit length in the norm l_1, and in the deterministic model it is always one of the basis states.

The query operation O_x maps the basis state $|i\rangle|a\rangle|z\rangle$ into the state $|i\rangle|(a+x_i) \bmod |\Sigma|\rangle|z\rangle$ (here we identify Σ with the residue classes $\bmod|\Sigma|$). Non-query operations are independent of x. A *k-query algorithm* is a sequence of $(k+1)$ operations $(U_0, U_1, \ldots, U_k)$ where U_i is unitary in the quantum and stochastic in the randomized model, and it is a permutation in the deterministic case. Initially the state of the computation is set to some fixed value $|0\rangle|0\rangle|0\rangle$, and then the sequence of operations $U_0, O_x, U_1, O_x, \ldots, U_{k-1}, O_x, U_k$ is applied. A quantum or randomized algorithm computes (with two-sided error) R if the observation of the appropriate last bits of the work register yield some $y \in T$ such that $(x,y) \in R$ with probability at least $2/3$. Then $\mathsf{QQC}(R)$ (resp. $\mathsf{RQC}(R)$) is the smallest k for which there exists a k-query quantum (resp. randomized) algorithm which computes R. In the case of deterministic algorithms of course exact computation is required, and the deterministic query complexity $\mathsf{DQC}(R)$ is defined then analogously. We have $\mathsf{DQC}(R) \geq \mathsf{RQC}(R) \geq \mathsf{QQC}(R)$.

Beals et al. [4] have shown that in the case of total functional oracle problems the deterministic and quantum complexities are polynomially related, and the gap is at most a degree 6 polynomial. For several partial functional problems exponential quantum speedups are known [10, 21].

In this paper we will give several results about Sperner problems in the black-box framework. In Section 5, we will prove that the deterministic query complexity of **REGULAR 2-SPM**, the black-box version of **2D-SPERNER** is $O(\sqrt{n})$. This matches the deterministic $\Omega(\sqrt{n})$ lower bound of Crescenzi and Silvestri [9]. The tightness of this bound was not known before. In fact, this result is the corollary of a general algorithm that solves the Sperner problems over pseudo-manifolds of arbitrary dimension. The complexity analysis of the algorithm will be expressed in **Theorem 4** in two combinatorial parameters of the pseudo-manifold: the size of its boundary and the separation number of its skeleton graph. In Section 6, we show that quantum, probabilistic, and deterministic query complexities of **REGULAR 2-SPM** are polynomially related. More precisely, in **Theorem 8** we will prove that its randomized complexity is $\Omega(\sqrt[4]{n})$ and that its quantum complexity is $\Omega(\sqrt[8]{n})$. This result is analogous to the polynomial relations obtained for the respective query complexities of PPP and PLS.

3 Mathematical background on simplicial complexes

For an undirected graph $G = (V, E)$, and for a subset $V' \subseteq V$ of the vertices, we denote by $G[V']$ the induced subgraph of G by V'. A graph $G'' = (V'', E'')$ is a subgraph of G, in notation $G'' \subseteq G$, if $V'' \subseteq V$ and $E'' \subseteq E$. The ring $\mathbb{Z}/(2)$ denotes the ring with 2 elements.

Definition 1 (Simplicial complex). A *simplicial complex* K is a non-empty collection of subsets of a finite set U, such that whenever $S \in K$ then $S' \in K$ for every $S' \subseteq S$. An element S of K of cardinality $d+1$ is called a *d-simplex*. A d'-simplex $S' \subseteq S$ is called a *d'-face of S*. We denote by K_d the set of d-simplices of K. An *elementary d-complex* is a simplicial complex that contains exactly one d-simplex and its subsets. The *dimension* of K, denoted by $\dim(K)$, is the largest d such that K contains a d-simplex.

The elements of K_0 are called the *vertices* of K, and the elements of K_1 are called the *edges* of K. The *skeleton graph* $G_K = (V_K, E_K)$ is the graph whose vertices are the vertices of K, and the edges are the edges of K.

Without loss of generality, we suppose that U consists of integers, and we identify $\{u\}$ with u, for $u \in U$.

Fact 1. *Let d be a positive integer. If S is an elementary d-complex, then G_S is the complete graph.*

Definition 2 (Oriented Simplex). For every positive integer n, we define an equivalence relation $\equiv_n$ over $\mathbb{Z}^n$, by $a \equiv_n b$ if there exists an even permutation σ such that $\sigma \cdot a = b$. For every $a \in \mathbb{Z}^n$ we denote by $[a]_{\equiv_n}$ the equivalence class of a for $\equiv_n$. The two equivalence classes of the orderings of the 0-faces of a simplex are called its *orientations*. An *oriented simplex* is a pair formed of a simplex and one of its orientations.

For an oriented d-simplex $(S, [\tau]_{\equiv_{d+1}})$, where τ is an ordering of the 0-faces of S, and a permutation σ over $\{1, \ldots, d+1\}$, we denote by $\sigma \cdot (S, [\tau]_{\equiv_{d+1}})$ the oriented d-simplex $(S, [\sigma \cdot \tau]_{\equiv_{d+1}})$. For every integer d, and every simplicial complex K whose simplices have been oriented, we denote by K_d the set of oriented d-simplices of K. From now on, S may denote an oriented or a non-oriented simplex. When S is an oriented simplex, $\bar{S}$ will denote the same simplex with the opposite orientation. We also define $S^{(i)}$ to be S if i is even, and to be $\bar{S}$ if i is odd. We will often specify an oriented simplex by an ordering of its 0-faces.

Definition 3. Let $S = (v_0, \ldots, v_d)$ be an oriented d-simplex. For every $0 \leq i \leq d$, for every $(d-1)$-face $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ of S, the *induced orientation* is the oriented $(d-1)$-simplex $(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d)^{(i)}$.

Definition 4. Let K be a simplicial complex whose simplices have been oriented, and let R be a ring. We define $C_d(K; R)$ as the submodule of the free R-module over the d-simplices of K with both possible orientations, whose elements are of the form $\sum_{S \in K_d}(c_S \cdot S + c_{\bar{S}} \cdot \bar{S})$, with $c_S \in R$, satisfying the relation $c_S = -c_{\bar{S}}$. The elements of $C_d(K; R)$ are called *d-chains*. For every oriented simplex S of K, we denote by $\langle S \rangle$ the element $S - \bar{S}$ of $C_d(K; R)$.

Let S be an oriented d-simplex $(v_0, v_1, \ldots v_d)$ of K. The *algebraic boundary* of $\langle S \rangle$, denoted by $\partial_d \langle S \rangle$, is the $(d-1)$-chain of $C_{d-1}(K; R)$ defined as $\partial_d \langle S \rangle = \sum_{i=0}^{d}(-1)^i \langle (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d) \rangle$.

Since $\partial_d \langle S \rangle = -\partial_d \langle \bar{S} \rangle$, the operator ∂_d has been correctly defined on a basis of $C_d(K; R)$ and can therefore be uniquely extended into a homomorphism $\partial_d : C_d(K; R) \to C_{d-1}(K; R)$. The proof of the next Lemma is straightforward.

Lemma 1. *Let S be an oriented d-simplex of a simplicial complex K. Denote by F_S the set of $(d-1)$-faces of S, and for every $S' \in F_S$ by $\tau_{S'}^S$ the induced orientation on S'. Then $\partial_d \langle S \rangle = \sum_{S' \in F_S} \langle (S', \tau_{S'}^S) \rangle$.*

Following an early version of a paper of Bloch [7], in the next definition we generalize the notion of pseudo-manifold, without the usual requirements of connectivity and pure dimensionality.

Definition 5. A simplicial complex $\mathcal{M}$ is a *pseudo d-manifold*, for a positive integer d, if

(i) $\mathcal{M}$ is a union of elementary d-complexes,

(ii) every $(d-1)$-simplex in $\mathcal{M}$ is a $(d-1)$-face of at most two d-simplices of $\mathcal{M}$.

The *boundary of $\mathcal{M}$* is the set of elementary $(d-1)$-complexes in $\mathcal{M}$ that belong exactly to one d-simplex of $\mathcal{M}$. We denote it by $\partial \mathcal{M}$. A pseudo d-manifold $\mathcal{M}$ is said to be *orientable* if it is possible to assign an orientation to each d-simplex of $\mathcal{M}$, such that for all $(d-1)$-simplex of $\mathcal{M}$ that is not on its boundary the orientations induced by the two d-simplices to which it belongs are opposite. Such a choice of orientations for all the d-simplices of $\mathcal{M}$ makes $\mathcal{M}$ *oriented*.

If the d-simplices of $\mathcal{M}$ are oriented, then there is a natural orientation of the $(d-1)$-simplices of $\partial\,\mathcal{M}$, where each $(d-1)$-simplex has the orientation induced by the oriented d-simplex of which it is a $(d-1)$-face. Notice that if $\mathcal{M}$ is a pseudo d-manifold, then $\partial\,M$ need not be a pseudo $(d-1)$-manifold. From now, all the simplicial complexes will be pseudo-manifolds. Observe that if $R = \mathbb{Z}/(2)$, then for any oriented d-simplex S, we have $\langle S \rangle = \langle \bar{S} \rangle$.

Definition 6. Given a simplicial complex K of dimension d, the *standard d-chain $\widehat{K}$ of K* will be defined depending on whether K is oriented as follows:

- if K is non-oriented, then $\widehat{K} = \sum_{S \in K_d} \langle (S, \tau_S) \rangle \in C_d(K, \mathbb{Z}/(2))$, for an arbitrary choice of orientations τ_S of the d-simplices S in K,

- if K is oriented, then $\widehat{K} = \sum_{S \in K_d} \langle (S, \tau_S) \rangle \in C_d(K, \mathbb{Z})$ where τ_S is the orientation of S in K.

Fact 2. *Let d be an integer, and let $\mathcal{M}$ be a pseudo d-manifold. Then,*

1. if $\mathcal{M}$ is not oriented the equality $\widehat{\partial\,\mathcal{M}} = \partial_d\,\widehat{\mathcal{M}}$ holds in $C_{d-1}(\partial\,\mathcal{M}, \mathbb{Z}/(2))$,

2. and if $\mathcal{M}$ is oriented the equality $\widehat{\partial\,\mathcal{M}} = \partial_d\,\widehat{\mathcal{M}}$ holds in $C_{d-1}(\partial\,\mathcal{M}, \mathbb{Z})$.

Proof. For every d-simplex S in $\mathcal{M}$, denote by F_S the set of $(d-1)$-faces of S, and for every $S' \in F_S$ denote by $\tau_{S'}^S$ the induced orientation of S' from S. From Lemma 1, we can write

$$\partial_d\,\widehat{\mathcal{M}} = \sum_{S \in \mathcal{M}_d} \partial_d\,\langle (S, \tau_S) \rangle = \sum_{S \in \mathcal{M}_d} \sum_{S' \in F_S} \langle (S', \tau_{S'}^S) \rangle.$$

From the definition of a pseudo d-manifold, we know that in the last sum each $(d-1)$-simplex that is not on the boundary of $\mathcal{M}$ appears exactly twice. If $\mathcal{M}$ is not oriented, then as the base ring is $\mathbb{Z}/(2)$, the only $(d-1)$-simplices that remain in the sum are those that are in $\partial\,\mathcal{M}$. If $\mathcal{M}$ is oriented, then from the definition of the orientability, it follows that each $(d-1)$-simplex that appears in two d-simplices of $\mathcal{M}$ appears in the sum once with each orientation. As for any oriented simplex S the equality $\langle S \rangle + \langle \bar{S} \rangle = 0$ holds, the only terms that do not cancel are the oriented $(d-1)$-simplices of the boundary. These $(d-1)$-simplices appear with the correct orientation. $\qquad\square$

4 Sperner Problems

Definition 7. Let K be a simplicial complex. A *labeling of K* is a mapping ℓ of the vertices of K into the set $\{0, \ldots, \dim(K)\}$. If a simplex S of K is labeled with all possible labels, then we say that S *is fully labeled*.

A labeling ℓ naturally maps every oriented d-simplex $S = (v_0, \ldots, v_d)$ to the equivalence class $\ell(S) = [(\ell(v_0), \ldots, \ell(v_d))]_{\equiv_{d+1}}$.

Definition 8. Given a labeling ℓ of a simplicial complex K, and an integer $0 \le d \le \dim(K)$, we define the *d-dimensional flow $N_d[\langle S \rangle]$* by

$$N_d[\langle S \rangle] = \begin{cases} 1 & \text{if } \ell(S) = [(0, 1, 2 \ldots, d)]_{\equiv_{d+1}}, \\ -1 & \text{if } \ell(S) = [(1, 0, 2, \ldots, d)]_{\equiv_{d+1}}, \\ 0 & \text{otherwise}, \end{cases}$$

and then extend it by linearity into a homomorphism $N_d : C_d(K; R) \to R$.

Sperner's Lemma [22] has been generalized in several ways. The following statement from [24] is also a straightforward consequence of results of [11].

Theorem 1 (Sperner's Lemma [22, 11, 24]). *Let K be a simplicial complex of dimension d, let ℓ be a labeling of K, and let R be a ring. For an element C of $C_d(K; R)$, we have $N_d[C] = (-1)^d N_{d-1}[\partial_d C]$.*

Using Fact 2, we translate Theorem 1 into terms of pseudo-manifolds.

Theorem 2 (Sperner's Lemma on pseudo-manifolds). *Let d be an integer, let $\mathcal{M}$ be a pseudo d-manifold, and let ℓ be a labeling of $\mathcal{M}$. Then $N_d[\widehat{\mathcal{M}}] = (-1)^d N_{d-1}[\widehat{\partial \mathcal{M}}]$ where*

$$\begin{cases} \widehat{\mathcal{M}} \in C_d(\mathcal{M}, \mathbb{Z}/(2)), \ \widehat{\partial \mathcal{M}} \in C_{d-1}(\partial \mathcal{M}, \mathbb{Z}/(2)), & \text{if } \mathcal{M} \text{ is not oriented,} \\ \widehat{\mathcal{M}} \in C_d(\mathcal{M}, \mathbb{Z}), \ \widehat{\partial \mathcal{M}} \in C_{d-1}(\partial \mathcal{M}, \mathbb{Z}), & \text{if } \mathcal{M} \text{ is oriented.} \end{cases}$$

This version of Sperner's lemma can be viewed, from a physicist's point of view, as a result equivalent to a global conservation law of a flow. If there is a source for the flow and the space is bounded then there must be a sink for that flow. More concretely, the lines of flow can be drawn over d-simplices, that goes from one d-simplex to another if they share a $(d-1)$-face that has all possible labels in $\{0, \ldots, d-1\}$. The sources and sinks of the flow are the fully labeled d-simplices. The lemma basically says that if the amount of flow entering the manifold at the boundary is larger than the exiting flow, then there must exist sinks inside. The local conservation is stated by the fact that if there is an ingoing edge, there will not be two outgoing edges, and conversely. Formally, we have the following.

Fact 3. *Let (S, τ_S) be an oriented d-simplex. Then at most two of its oriented $(d-1)$-faces have a non-zero image by N_{d-1}. Moreover, if there are exactly two $(d-1)$-faces $(S', \tau_{S'}^S)$ and $(S'', \tau_{S''}^S)$ that have non-zero image by N_{d-1}, then $N_d[\langle(S, \tau_S)\rangle] = 0$ and $N_{d-1}[\langle(S', \tau_{S'}^S)\rangle] = -N_{d-1}[\langle(S'', \tau_{S''}^S)\rangle]$.*

This gives a relation between the problem of finding fully labeled d-simplices and the natural complete problems for the parity argument classes. We can consider an oriented d-simplex (S, τ_S) with $N_d[\langle(S, \tau_S)\rangle] = 1$ as a source for the flow, and $(S', \tau_{S'})$ with $N_d[\langle(S', \tau_{S'})\rangle] = -1$ as a sink.

We now state the black-box Sperner problems we will consider.

Sperner on Pseudo d-Manifolds (d-SPM)

Input:	a pseudo d-manifold $\mathcal{M}$, and $S \in \mathcal{M}_d$.
Oracle input:	a labeling $\ell : \mathcal{M}_0 \to \{0, 1, \ldots, d\}$.
Promise:	one of the two conditions holds:
	a) $N_{d-1}[\widehat{\partial \mathcal{M}}] = 1$ in $C_{d-1}(\partial \mathcal{M}, \mathbb{Z}/(2))$,
	b) $N_{d-1}[\widehat{\partial \mathcal{M}}] = 0$ in $C_{d-1}(\partial \mathcal{M}, \mathbb{Z}/(2))$ and $N_d[\langle S \rangle] = 1$ in $C_d(\mathcal{M}, \mathbb{Z}/(2))$.
Output:	$S' \in \mathcal{M}_d$ such that $N_d[\langle S' \rangle] = 1$, with $S \neq S'$ for case b.

Oriented Sperner on Pseudo d-Manifolds (d-OSPM)

Input:	an oriented pseudo d-manifold $\mathcal{M}$, and $S \in \mathcal{M}_d$.
Oracle input:	a labeling $\ell : \mathcal{M}_0 \to \{0, 1, \ldots, d\}$.
Promise:	one of the two conditions holds:
	a) $(-1)^d N_{d-1}[\widehat{\partial \mathcal{M}}] < 0$ in $C_{d-1}(\partial \mathcal{M}, \mathbb{Z})$,
	b) $(-1)^d N_{d-1}[\widehat{\partial \mathcal{M}}] = 0$ in $C_{d-1}(\partial \mathcal{M}, \mathbb{Z})$ and $N_d[\langle S \rangle] = 1$ in $C_d(\mathcal{M}, \mathbb{Z})$.
Output:	$S' \in \mathcal{M}_d$ such that $N_d[\langle S' \rangle] = -1$.

We will deal in particular with the following important special case of 2-**SPM**. Let $V_m = \{(i, j) \in \mathbb{N}^2 \mid 0 \le i + j \le m\}$. Observe that $|V_m| = \binom{m+2}{2}$.

Regular Sperner (REGULAR 2-SPM)

Input:	$n = \binom{m+2}{2}$ for some integer m.
Oracle input:	a labeling $\ell : V_m \to \{0, 1, 2\}$.
Promise:	for $0 \le k \le m$, $\ell(0, k) \ne 1$, $\ell(k, 0) \ne 0$, and $\ell(k, m - k) \ne 2$.
Output:	p, p' and $p'' \in V$, such that $p' = p + (\varepsilon, 0)$, $p'' = p + (0, \varepsilon)$ for some $\varepsilon \in \{-1, 1\}$, and $\{\ell(p), \ell(p'), \ell(p'')\} = \{0, 1, 2\}$.

In fact, **REGULAR** 2-**SPM** on input $n = \binom{m+2}{2}$ is the instance of d-**SPM** on the regular m-subdivision of an elementary 2-simplex. Theorem 2 states that both d-SPM and d-OSPM have always a solution. The solution is not necessarily unique as it can be easily checked on simple instances. Thus the problems are not functional oracle problems.

5   Black-box algorithms for pseudo d-manifolds

The purpose of this section is to give a black-box algorithm for d-**SPM** and d-**OSPM**. To solve these problems, we adopt a divide and conquer approach. This kind of approach was successfully used in [15, 14] and [19], to study the query complexity of the oracle version of the Local Search problem. However, the success of the divide and conquer paradigm for Sperner problems relies heavily on the use of the very strong statement of Sperner's Lemma that is given in Theorem 2. The usual, simpler version of Sperner's Lemma, like the one given in [18] does not appear to be strong enough for this purpose. Observe that though the standard proof of Sperner's Lemma is constructive, it yields only an algorithm of complexity $O(n)$. In our algorithms the division of the pseudo d-manifold $\mathcal{M}$ will be done according to the combinatorial properties of its skeleton graph. The particular parameter we will need is its *iterated separation number* that we introduce now for general graphs.

Definition 9. Let $G = (V, E)$ be a graph. If A and C are subsets of V such that $V = A \cup C$, and that there is no edge between $A \setminus C$ and $C \setminus A$, then (A, C) is said to be a *separation* of the graph G, in notation $(A, C) \prec G$. The set $A \cap C$ is called a *separator* of the graph G.

The *iterated separation number* is defined by induction on the size of the graph G by $s(G) = \min_{(A,C) \prec G} \{|A \cap C| + \max(s(G[A \setminus C]), s(G[C \setminus A]))\}$. A pair $(A, C) \prec G$ such that $s(G) = |A \cap C| + \max(s(G[A \setminus C]), s(G[C \setminus A]))$ is called a *best separation* of G.

The iterated separation number of a graph is equal to the *value of the separation game* on the graph G, which was introduced in [15]. In that article, that value was defined as the gain of a player in a certain game. Notice, also, that the iterated separation number is at most $\log |V|$ times the *separation number* as defined in [19]. Before giving the algorithms, and their analyses, we still need a few observations.

Lemma 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two pseudo d-manifolds, such that $\mathcal{A} \cup \mathcal{B}$ is also a pseudo d-manifold. Let ℓ be a labeling of $\mathcal{A} \cup \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ have no d-simplex in their intersection, then $N_d[\widehat{\mathcal{A} \cup \mathcal{B}}] = N_d[\widehat{\mathcal{A}}] + N_d[\widehat{\mathcal{B}}]$.*

Lemma 3. *Let $\mathcal{M}$ be a pseudo d-manifold, and $\mathcal{M}'$ be a union of elementary d-complexes such that $\mathcal{M}' \subseteq \mathcal{M}$. Then $\mathcal{M}'$ is a pseudo d-manifold.*

Theorem 3. *Let $\mathcal{M}$ be a pseudo d-manifold, H a subset of $\mathcal{M}_0$, and ℓ be a labeling of the vertices of $\mathcal{M}$. Let $(A, C) \prec G_{\mathcal{M}}[\mathcal{M}_0 \setminus H]$, $B = H \cup (A \cap C)$, and $M' = A \setminus C$ and $M'' = C \setminus A$. Denote by $\mathcal{B}$ the set of elementary d-complexes of $\mathcal{M}$ whose vertices are all in B, and by $\mathcal{M}'$ (resp. $\mathcal{M}''$) the set of elementary d-complexes of which at least one of the vertices belongs to M' (resp. M''). Denote also by $\mathcal{B}'$ the set of elementary $(d-1)$-complexes of $\mathcal{M}$ whose vertices are all in B. Then,*

(i) $\mathcal{B}$, $\mathcal{M}'$, $\mathcal{M}''$ and $\mathcal{M}' \cup \mathcal{M}''$ are pseudo d-manifolds,

(ii) if $H \neq \mathcal{M}_0$ then $\mathcal{B}$, $\mathcal{M}'$ and $\mathcal{M}''$ are proper subsets of $\mathcal{M}$,

(iii) $N_d[\widehat{\mathcal{M}}] = N_d[\widehat{\mathcal{B}}] + N_d[\widehat{\mathcal{M}'}] + N_d[\widehat{\mathcal{M}''}]$,

(iv) the inclusions $\partial \mathcal{M}' \subseteq (\partial \mathcal{M}) \cup \mathcal{B}'$ and $\partial \mathcal{M}'' \subseteq (\partial \mathcal{M}) \cup \mathcal{B}'$ hold,

Proof. Clearly, the complexes $\mathcal{B}$, $\mathcal{M}'$, $\mathcal{M}''$ and $\mathcal{M}' \cup \mathcal{M}''$ are pseudo d-manifolds, according to the previous lemma.

For *(ii)*, assume $H \neq \mathcal{M}_0$. Let x be a vertex in $A \setminus C$ and S be a d-simplex that contains it. By Fact 1, all the points of S are neighbors of x in the skeleton graph. The neighbors of x are all in A, since there is no edge between $A \setminus C$ and $C \setminus A$. Therefore, S is in $\mathcal{M}'$ but not in $\mathcal{B} \cup \mathcal{M}''$. By symmetry of $\mathcal{M}'$ and $\mathcal{M}''$, there is also a d-simplex which is not in $\mathcal{M}'$.

Let us now turn to prove *(iii)*. Let S be an elementary d-complex in $\mathcal{M}$. As B separates $G_{\mathcal{M}}[\mathcal{M}_0 \setminus H]$ into the two components M' and M'', it is not possible that S contains elements from both M' and from M'', as from Fact 1 we know that G_S is a complete graph. So, either all the vertices of S are in B, or in $B \cup M'$, or they are in $B \cup M''$. This proves that S belongs to $\mathcal{B} \cup \mathcal{M}' \cup \mathcal{M}''$. Therefore $\mathcal{M} \subseteq \mathcal{B} \cup \mathcal{M}' \cup \mathcal{M}''$. The converse inclusion clearly holds, which implies that it is in fact an equality. Moreover, from their definitions, the simplicial complexes $\mathcal{B}$, $\mathcal{M}'$ and $\mathcal{M}''$ have no d-simplex in common. Then using the first point, two applications of Lemma 2 allow us to deduce the announced equality in *(iii)*.

For *(iv)*, let S be a $(d-1)$-simplex in $\partial \mathcal{M}'$. It is not in $\partial \mathcal{M}$ if and only if it belongs to two d-simplices of $\mathcal{M}$. We will prove that if S belongs to two d-simplices of $\mathcal{M}$, then its 0-faces must all lie in B. Assume that S is a $(d-1)$-face common to two d-simplices T_1 and T_2. We can assume without loss of generality that T_1 belongs to $\mathcal{M}'$. But T_2 can not be in $\mathcal{M}'$ as S is in the boundary of $\mathcal{M}'$. So, either T_2 is in $\mathcal{B}$, or it is in $\mathcal{M}''$. In the first case, it immediately follows that S has all its 0-faces in B, as it is a face of a d-simplex whose 0-faces all lie in B. In the second case, again, the only possibility is that S has all its 0-faces in B, as else a vertex in M' and a vertex in M'' would be neighbors. This proves the third point for $\partial \mathcal{M}'$. The proof is the same for $\partial \mathcal{M}''$. $\square$

We are now ready to state Algorithm 1 and Algorithm 2 which solve respectively d-**SPM** and d-**OSPM** when the labels of the 0-faces of $\partial \mathcal{M}$ are also known. We next give the result which states the correctness of our algorithms and specifies their complexities.

Lemma 4. *If $\mathcal{M}$ and S satisfy the promises of the respective Sperner problems, then Algorithms 1 and 2 return a solution and use at most $s(G_{\mathcal{M}}[\mathcal{M}_0 \setminus H])$ queries.*

Algorithm 1 Main routine for solving d-**SPM**.

Input: A pseudo d-manifold $\mathcal{M}$, $S \in \mathcal{M}_d$, a set $H \supseteq (\partial\mathcal{M})_0$ together with the labels of its elements.

Let $(A, C) \prec G_\mathcal{M}[\mathcal{M}_0 \setminus H]$ be a best separation, and $B = H \cup (A \cap C)$.
Let the complexes $\mathcal{B}$, $\mathcal{M}'$ and $\mathcal{M}''$ be defined as in Theorem 3.
Query the labels of the vertices in $A \cap C$.
if $\mathcal{B}$ contains a fully labeled elementary d-complex **then**
    Return the corresponding oriented d-simplex.
end if
Evaluate $N_{d-1}[\widehat{\partial\,\mathcal{B}}]$, $N_{d-1}[\widehat{\partial\,\mathcal{M}'}]$ and $N_{d-1}[\widehat{\partial\,\mathcal{M}''}]$.
if $N_{d-1}[\widehat{\partial\,K}] = 1$ for $K \in \{\mathcal{B}, \mathcal{M}', \mathcal{M}''\}$ **then**
    Iterate on K, any d-simplex $S \in K$, and B with the labels of its elements.
else
    Iterate on $K \in \{\mathcal{B}, \mathcal{M}', \mathcal{M}''\}$ containing S, S and B with the labels of its elements.
end if

Algorithm 2 Main routine for solving d-**OSPM**.

Input: A pseudo d-manifold $\mathcal{M}$, $S \in \mathcal{M}_d$, a set $H \supseteq (\partial\mathcal{M})_0$ together with the labels of its elements.

Let $(A, C) \prec G_\mathcal{M}[\mathcal{M}_0 \setminus H]$ be a best separation, and $B = H \cup (A \cap C)$.
Let the complexes $\mathcal{B}$, $\mathcal{M}'$ and $\mathcal{M}''$ be defined as in Theorem 3.
Query the labels of the vertices in $A \cap C$.
if $\mathcal{B}$ contains a fully labeled elementary d-complex **then**
    Return the corresponding oriented d-simplex.
end if
Evaluate $N_{d-1}[\widehat{\partial\,\mathcal{B}}]$, $N_{d-1}[\widehat{\partial\,\mathcal{M}'}]$ and $N_{d-1}[\widehat{\partial\,\mathcal{M}''}]$.
if $(-1)^d N_{d-1}[\widehat{\partial\,K}] < 0$ on $K \in \{\mathcal{B}, \mathcal{M}', \mathcal{M}''\}$ **then**
    Iterate the algorithm on K, any d-simplex $S \in K$, and B with the labels of its elements.
else
    Iterate the algorithm on $K \in \{\mathcal{B}, \mathcal{M}', \mathcal{M}''\}$ containing S, S and B with the labels of its elements.
end if

Proof. We will prove the two claims for Algorithm 1 by induction, the proofs for Algorithm 2 are similar.

We start by proving the correctness. First observe that there is always enough information for the evaluations of the flows. Indeed by *(iv)* of Theorem 3, all the 0-faces of $\partial\,\mathcal{B}$, $\partial\,\mathcal{M}'$ and $\partial\,\mathcal{M}''$ are in $\partial\,\mathcal{M} \cup B$. The labels of the 0-faces of $\mathcal{M}$ are given as an input, and the labels of B are queried right before the flow evaluations.

Let us now consider an input that satisfies the promise of d-**SPM**, and where the number of d-simplices in $\mathcal{M}$ is n. If $n = 1$, then $\mathcal{M} = \mathcal{B}$ is an elementary d-complex, and by the promise, it is fully labeled. Therefore, the output of the algorithm is correct. When $n > 1$, we will prove that the recursive call will be made on an input which also satisfies the promise, and where the number of d-simplices in the pseudo d-manifold is less than n.

From Theorem 1 and *(iii)* of Theorem 3, we have $N_{d-1}[\widehat{\partial\,\mathcal{M}}] = N_{d-1}[\widehat{\partial\,\mathcal{B}}] + N_{d-1}[\widehat{\partial\,\mathcal{M}'}] + N_{d-1}[\widehat{\partial\,\mathcal{M}''}]$. In case *a)* of the promise, this sum is equal to 1, and therefore there exists a $K \in \{\mathcal{B}, \mathcal{M}', \mathcal{M}''\}$ for which $N_{d-1}[\widehat{\partial\,K}] = 1$. In case *b)* of the promise, either there exists a $K \in \{\mathcal{B}, \mathcal{M}', \mathcal{M}''\}$ for which $N_{d-1}[\widehat{\partial\,K}] = 1$, or there exists a $K \in \{\mathcal{B}, \mathcal{M}', \mathcal{M}''\}$ for which $N_{d-1}[\widehat{\partial\,K}] = 0$ and $S \in K$. In both cases, the number of d-simplices in K is less than the number of d-simplices of $\mathcal{M}$ because of *(ii)* of Theorem 3.

Let us now prove the bound on the complexity. For every pseudo d-manifold $\mathcal{M}$, denote by $T(\mathcal{M}, H)$ the number of queries made by the algorithm on $\mathcal{M}$ with the set H of labels. Each recursive call ends in three possible ways:

    1) it stops after the first test if $\mathcal{M}$ is a fully labeled elementary d-complex,

    2) it iterates on $\mathcal{B}$,

    3) or it iterates on $\mathcal{M}'$ or on $\mathcal{M}''$.

In the first case, no queries are made, as all vertices of $\mathcal{M}$ are on its boundary. In the second case, $|A \cap C|$ queries are made, as in further iterations all labels will be known. In the third case, the number of queries is at most $|A \cap C| + \max(T(\mathcal{M}', B), T(\mathcal{M}'', B))$. Thus, we get $T(\mathcal{M}, H) \leq |A \cap C| + \max(T(\mathcal{M}', B), T(\mathcal{M}'', B))$.

We now prove that $T(\mathcal{M}, H) \leq s(G_{\mathcal{M}}[\mathcal{M}_0 \setminus H])$ for every pseudo d-manifold $\mathcal{M}$ and set of vertices $H \subseteq \mathcal{M}_0$. If $\mathcal{M}$ is an elementary d-complex, then the algorithm does not make any query, and therefore $T(\mathcal{M}, H) = 0$, and the statement is trivial.

Let now $\mathcal{M}$ be a pseudo d-manifold that is not an elementary d-complex. We have $G_{\mathcal{M}'}[\mathcal{M}'_0 \setminus B] \subseteq G_{\mathcal{M}}[M'] = G_{\mathcal{M}}[A \setminus C]$ and $G_{\mathcal{M}''}[\mathcal{M}''_0 \setminus B] \subseteq G_{\mathcal{M}}[M''] = G_{\mathcal{M}}[C \setminus A]$, for a best separation $(A, C) \prec G_{\mathcal{M}}[\mathcal{M}_0 \setminus H]$. Using the induction hypothesis, we get $T(\mathcal{M}', B) \leq s(G_{\mathcal{M}'}[\mathcal{M}'_0 \setminus B])$ and $T(\mathcal{M}'', B) \leq s(G_{\mathcal{M}''}[\mathcal{M}''_0 \setminus B])$. Since $s(G') \leq s(G)$ if G' is a subgraph of G, we get $T(\mathcal{M}', B) \leq s(G_{\mathcal{M}}[A \setminus C])$ and $T(\mathcal{M}'', B) \leq s(G_{\mathcal{M}}[C \setminus A])$. As (A, C) is a best separation of $G_{\mathcal{M}}[\mathcal{M}_0 \setminus H]$, this proves the inequality $T(\mathcal{M}, H) \leq s(G_{\mathcal{M}}[\mathcal{M}_0 \setminus H])$. $\square$

Theorem 4. $\mathsf{DQC}(d\text{-}\mathbf{SPM}) = O(s(G_{\mathcal{M}}[\mathcal{M}_0 \setminus (\partial\,\mathcal{M})_0])) + |(\partial\,\mathcal{M})_0|$ *and*
$\mathsf{DQC}(d\text{-}\mathbf{OSPM}) = O(s(G_{\mathcal{M}}[\mathcal{M}_0 \setminus (\partial\,\mathcal{M})_0])) + |(\partial\,\mathcal{M})_0|$.

Proof. The algorithms consist in querying the labels of the vertices of $\partial\mathcal{M}$ and then running respectively Algorithm 1 or Algorithm 2 with the initial choice $H = (\partial\,\mathcal{M})_0$. $\square$

To bound the complexity of our algorithms we need an upper-bound on the iterated separator number of the skeleton graph. The following theorem gives, for any graph, an upper bound on the size of a balancing separator, whose deletion leaves the graph with two roughly equal size components. The bound depends on the genus and the number of vertices of the graph.

Theorem 5 (Gilbert, Hutchinson, Tarjan [12]). *A graph of genus g with n vertices has a set of at most $6\sqrt{g \cdot n} + 2\sqrt{2n} + 1$ vertices whose removal leaves no component with more than $2n/3$ vertices.*

For our purposes we can immediately derive an upper bound on the iterated separation number.

Corollary 1. *For graphs $G = (V, E)$ of size n and genus g we have $s(G) \leq \lambda(6\sqrt{g \cdot n} + 2\sqrt{2n}) + \log_{3/2} n$, where λ is solution of $\lambda = 1 + \lambda\sqrt{2/3}$.*

Proof. Let us prove this fact by induction over n. It obviously holds for $n = 1$. Assume now that $n > 1$. Theorem 5 shows that there exist three pairwise disjoint sets S_1, S_2 and S_3 such that

$V = S_1 \cup S_2 \cup S_3$, $|S_2| \le 6\sqrt{g \cdot n} + 2\sqrt{2n} + 1$ and $|S_1|, |S_3| \le 2n/3$. If we let $A = S_1 \cup S_2$ and $C = S_2 \cup S_3$, then $(A, C) \prec G$ and $A \cap C = S_2$. The construction implies that $|A \setminus C|, |C \setminus A| \le 2n/3$. Using the induction hypothesis, we get

$$
\begin{aligned}
s(G) &\le |A \cap C| + \max(s(G[A \setminus C]), s(G[C \setminus A])) \\
&\le 6\sqrt{g \cdot n} + 2\sqrt{2n} + 1 + \lambda(6\sqrt{g \cdot 2n/3} + 2\sqrt{2 \cdot 2n/3}) + \log_{3/2}(2n/3) \\
&\le \lambda(6\sqrt{g \cdot n} + 2\sqrt{2n}) + \log_{3/2} n.
\end{aligned}
$$

$\square$

In general, there is no immediate relationship between the genus of a pseudo d-manifold and the genus of its skeleton graph. However, if the pseudo d-manifold $\mathcal{M}$ is a triangulated oriented surface, then the genus of the graph is equal to the genus of $\mathcal{M}$. Used in conjunction with Corollary 1, Theorem 4 gives an effective upper bound for pseudo d-manifolds.

Corollary 2. *Let $\mathcal{M}$ be a pseudo d-manifold such that $G_{\mathcal{M}}$ is of size n and of genus g. Then, $\mathsf{DQC}(d\text{-}\mathbf{SPM}) = O(\sqrt{g}) \cdot \sqrt{n} + |(\partial\mathcal{M})_0|$ and $\mathsf{DQC}(d\text{-}\mathbf{OSPM}) = O(\sqrt{g}) \cdot \sqrt{n} + |(\partial\mathcal{M})_0|$.*

Since the skeleton graph of the underlying pseudo 2-manifold of **REGULAR** 2-**SPM** is planar, it has genus 0. Thus we get:

Theorem 6. $\mathsf{DQC}(\mathbf{REGULAR}\ 2\text{-}\mathbf{SPM}) = O(\sqrt{n})$.

In the next section, we show nontrivial lower bounds on the randomized and the quantum query complexity of the **REGULAR** 2-**SPM** problem. Observe that for some general instances of the 2-**SPM** over the same pseudo 2-manifold we can easily derive exact lower bounds from the known complexity of Grover's search problem [6]. For example, if a labeling is 2 everywhere, except on two consecutive vertices on the boundary where it takes respectively the values 0 and 1, then finding a fully labeled 2-simplex is of the same complexity as finding a distinguished element on the boundary.

6   Lower bounds for REGULAR 2-SPM

We denote by **UNIQUE-SPERNER** all those instances of **REGULAR** 2-**SPM** for which there exists a unique fully labeled triangle. There exist several equivalent adversary methods for proving quantum lower bounds in the query model [23]. Here, we will use the weighted adversary method [2, 3, 13].

Theorem 7. *Let Σ be a finite set, let $n \ge 1$ be an integer, and let $S \subseteq \Sigma^n$ and S' be sets. Let $f : S \to S'$. Let Γ be an arbitrary $S \times S$ nonnegative symmetric matrix that satisfies $\Gamma[x, y] = 0$ whenever $f(x) = f(y)$. For $1 \le k \le n$, let Γ_k be the matrix such that $\Gamma_k[x, y] = 0$ if $x_k = y_k$, and $\Gamma_k[x, y] = \Gamma[x, y]$ otherwise. For all $S \times S$ matrix M and $x \in S$, let $\sigma(M, x) = \sum_{y \in S} M[x, y]$. Then*

$$
\mathsf{QQC}(f) = \Omega\left(\min_{\Gamma[x,y] \neq 0, x_k \neq y_k} \sqrt{\frac{\sigma(\Gamma, x)\sigma(\Gamma, y)}{\sigma(\Gamma_k, x)\sigma(\Gamma_k, y)}}\right),
$$

$$
\mathsf{RQC}(f) = \Omega\left(\min_{\Gamma[x,y] \neq 0, x_k \neq y_k} \max\left(\frac{\sigma(\Gamma, x)}{\sigma(\Gamma_k, x)}, \frac{\sigma(\Gamma, y)}{\sigma(\Gamma_k, y)}\right)\right).
$$

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For the lower bound we will consider specific instances of **REGULAR 2-SPM**. For that, we need a few definitions. For any binary sequence b, let $|b|$ denote the length of the sequence b, and for $i = 0, 1$ let $w_i(b)$ be the number of bits i in b. For $0 \le t \le |b|$, let $b^t = b_1 \dots b_t$ denote the prefix of length t of b.

The instances of **REGULAR 2-SPM** we will consider are those whose oracle inputs C_b are induced by binary sequences $b = b_1 \dots b_{m-2}$ of length $m - 2$ as follows:

$$
C_b(i,j) = \begin{cases}
1 & \text{if } j = 0 \text{ and } i \neq 0, \\
2 & \text{if } i = 0 \text{ and } j \neq m, \\
0 & \text{if } i + j = m \text{ and } j \neq 0, \\
1 & \text{if there exists } 0 \le t \le m - 2 \text{ with } (i,j) = (w_0(b^t) + 1, w_1(b^t)), \\
2 & \text{if there exists } 0 \le t \le m - 2 \text{ with } (i,j) = (w_0(b^t), w_1(b^t) + 1), \\
0 & \text{otherwise.}
\end{cases}
$$

Notice that the first and fourth (*resp.* second and fifth) conditions can be simultaneously satisfied, but the labeling definition is consistent. Also observe that, for any b, there is a unique fully labeled triangle, whose coordinates are $\{(w_0(b) + 1, w_1(b)), (w_0(b), w_1(b) + 1), (w_0(b) + 1, w_1(b) + 1)\}$. Therefore C_b is an instance of **UNIQUE-SPERNER**. We illustrate an instance of C_b in Figure 1.

It turns out that technically it will be easier to prove the lower bound for a problem which is closely related to the above instances of **REGULAR 2-SPM**, that we call **SNAKE**. Recall that $V_m = \{(i,j) \in \mathbb{N}^2 \mid 0 \le i + j \le m\}$. For every binary sequence $b = b_1 \dots b_{m-2}$, we denote by O_b the function $V_m \to \{0, 1\}$ defined for $p \in V_m$ by

$$
O_b(p) = \begin{cases}
1 & \text{if there exists } 0 \le t \le m - 2 \text{ with } (i,j) = (w_0(b^t) + 1, w_1(b^t)), \\
0 & \text{otherwise.}
\end{cases}
$$

See again Figure 1 for an example.

SNAKE

Input:	$n = \binom{m}{2}$ for some integer m.
Oracle input:	a function $f : V_m \to \{0, 1\}$.
Promise:	there exists a binary sequence $b = b_1 \dots b_{m-2}$ such that $f = O_b$.
Output:	$(w_0(b), w_1(b))$.

We recall here the definition of [19] of c-query reducibility between black-box problems, which we will use to prove our lower bound.

Definition 10. For an integer $c > 0$, a functional oracle problem $A : S_1 \to T_1$ with $S_1 \subseteq \Sigma_1^n$ is *c-query reducible* to a functional oracle problem $B : S_2 \to T_2$ with $S_2 \subseteq \Sigma_2^{n'}$ if the following two conditions hold:

(i) $\exists \alpha : S_1 \to S_2, \quad \exists \beta : T_2 \to T_1$, such that $\forall x \in S_1, \ A(x) = \beta(B(\alpha(x)))$,

(ii) $\exists \gamma_1, \dots, \gamma_c : \{1, \dots, n'\} \to \{1, \dots, n\}$ and $\gamma : \{1, \dots, n'\} \times \Sigma_1^c \to \Sigma_2$ such that $\forall x \in S_1, k \in \{1, \dots, n'\}, \quad \alpha(x)(k) = \gamma(k, x_{\gamma_1(k)}, \dots, x_{\gamma_c(k)})$.

Lemma 5 ([19]). *If A is c-query reducible to B then $\mathsf{QQC}(B) \ge \mathsf{QQC}(A)/2c$, and $\mathsf{RQC}(B) \ge \mathsf{RQC}(A)/c$.*
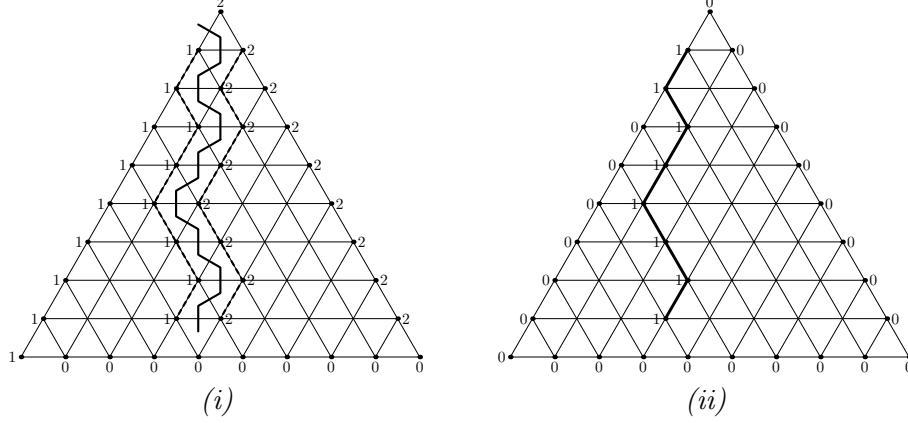
Figure 1: In the coordinates system of the Figure, the point $(0,0)$ is the highest corner of the triangles, the x coordinates increase by going down and left, and the y coordinates increase by going down and right. On sub-figure *(i)*, the labeling C_b corresponds to the binary sequence $b = 0100110$. On sub-figure *(ii)*, the labeling O_b corresponds to the same sequence b. The unmarked vertices are all labeled 0.

Lemma 6. SNAKE *is 3-query reducible to* **UNIQUE-SPERNER**.

Proof. We define the oracle transformations as $\alpha(O_b) = C_b$, and

$$\beta(\{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}) = (\min\{i_1, i_2, i_3\}, \min\{j_1, j_2, j_3\}).$$

Obviously, α and β satisfy the first condition of the definition.

We now turn to the simulation of an oracle for **UNIQUE-SPERNER** by an oracle for **SNAKE**. If the query concerns a point (i, j) on the boundary, the answer is independent from the oracle, and is given according to the definition of C_b, for any b. Otherwise, the simulator will query the point and its left and right neighbors in the sense of the Figure 1, from which the value of C_b can be easily determined. Formally, for such a point (i, j), let the functions γ, γ_1, γ_2 and γ_3 be defined as

$$\gamma((i, j), a_1, a_2, a_3) = \begin{cases} 1 & \text{if } (a_1, a_2, a_3) = (0, 1, 0), \\ 2 & \text{if } (a_1, a_2, a_3) = (1, 0, 0), \\ 0 & \text{otherwise,} \end{cases}$$

and $\gamma_1(i, j) = (i+1, j-1)$, $\gamma_2(i, j) = (i, j)$, $\gamma_3(i, j) = (i-1, j+1)$. $\qquad\square$

Lemma 7. *The query complexity of* **SNAKE** *f satisfies*

$$\mathsf{RQC}(\textbf{SNAKE}) = \Omega(\sqrt[4]{n}) \ \text{and} \ \mathsf{QQC}(\textbf{SNAKE}) = \Omega(\sqrt[8]{n}).$$

Proof. We give now the definition of the adversary matrix Γ which will be a $2^{m-2} \times 2^{m-2}$ symmetric matrix, whose rows and columns will be indexed by the labelings O_b, when $b \in \{0, 1\}^{m-2}$. For the sake of simplicity, we will only use binary sequences to denote rows and columns, instead of the induced labelings. For two binary sequences b and b', we denote by $b \wedge b'$ their longest common

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prefix. Then let

$$\Gamma[b, b'] = \begin{cases} 0 & \text{if } w_0(b) = w_0(b'), \\ 2^{|b \wedge b'|} & \text{otherwise.} \end{cases}$$

For a given binary sequence b, there are $2^{m-2-(d+1)}$ sequences that have longest common prefix of length d with b. Out of them, $\binom{m-3-d}{w_{b_{d+1}}(b)-w_{b_{d+1}}(b^d)}$ will give the same output as b. Therefore,

$$\sigma(\Gamma, b) = \sum_{d=0}^{m-4} 2^d \left[2^{m-3-d} - \binom{m-3-d}{w_{b_{d+1}}(b) - w_{b_{d+1}}(b^d)} \right]$$

$$\geq (m-3)2^{m-3} - \sum_{d=0}^{m-4} 2^d \binom{m-3-d}{\lfloor (m-3-d)/2 \rfloor}$$

$$\geq (m-3)2^{m-3} - \left(\sum_{d=0}^{m-4} \frac{2^{m-3}}{\sqrt{m-3-d}} \right)$$

$$\geq (m-3)2^{m-3} - O(\sqrt{m}2^m) = \Omega(m2^m).$$

We now turn to bound from above $\sigma(\Gamma_p, b)$ and $\sigma(\Gamma_p, b')$ when $\Gamma[b, b'] \neq 0$ and $O_b(p) \neq O_{b'}(p)$. Let us now fix a point $p = (i, j)$ with $1 \leq i + j \leq m - 1$, and two sequences b and b', such that $O_b(p) \neq O_{b'}(p)$. We assume that $O_b(p) = 0$ and $O_{b'}(p) = 1$. We trivially upper bound $\sigma(\Gamma_p, b')$ by $\sigma(\Gamma, b') = O(m2^m)$.

We will now upper-bound $\sigma(\Gamma_p, b)$. Set $h = i + j$. If a sequence b'' is such that $O_{b''}(p) = 1$, then the length of its longest common prefix with b is at most $h - 2$. We regroup these sequences according to the value $|b \wedge b''|$. The number of sequences b'' for which $|b \wedge b''| = d$ and $O_{b''}(p) = 1$ is at most $\binom{h-1-(d+1)}{\lfloor h-1-(d+1))/2 \rfloor}2^{m-h-1}$. Therefore we can bound $\sigma(\Gamma_p, b)$ as

$$\sigma(\Gamma_p, b) \leq \sum_{d=0}^{h-2} 2^d \cdot \binom{h-d-2}{\lfloor (h-d-2)/2 \rfloor} 2^{m-h-1}$$

$$\leq \sum_{d=0}^{h-2} \left[2^{-(h-d-2)} \cdot \binom{h-d-2}{\lfloor (h-d-2)/2 \rfloor} \right] 2^{m-3} = O(\sqrt{m}2^m).$$

By Theorem 7 we conclude that

$$\mathsf{RQC}(\textbf{SNAKE}) = \Omega\left(\max\left(\frac{m2^m}{m2^m}, \frac{m2^m}{\sqrt{m}2^m} \right) \right) = \Omega(\sqrt[4]{n}),$$

$$\mathsf{QQC}(\textbf{SNAKE}) = \Omega\left(\frac{m2^m}{\sqrt{m2^m \cdot \sqrt{m}2^m}} \right) = \Omega(\sqrt[8]{n}).$$

$\square$

Theorem 8. *The query complexity of* **REGULAR** 2-**SPM** *satisfies*

$$\mathsf{RQC}(\textbf{REGULAR } 2\text{-}\textbf{SPM}) = \Omega(\sqrt[4]{n}) \text{ and } \mathsf{QQC}(\textbf{REGULAR } 2\text{-}\textbf{SPM}) = \Omega(\sqrt[8]{n}).$$

Proof. By Lemma 5 and 6, the lower bounds of Lemma 7 for **SNAKE** also apply to **REGULAR** 2-**SPM**. $\square$

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